Non-Zero-Sum Stochastic Differential Games of Controls and Stoppings

Qinghua Li

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Based on two preprints:

- Martingale Interpretation to a Non-Zero-Sum Stochastic Differential Game of Controls and Stoppings
  I. Karatzas, Q. Li, 2009
- A BSDE Approach to Non-Zero-Sum Stochastic Differential Games of Controls and Stoppings
  I. Karatzas, Q. Li, S. Peng, 2009

Bibliography

Non-Zero-Sum Game and Nash Equilibrium

John F. Nash (1949):

One may define a concept of **AN *n*-PERSON GAME** in which each player has a finite set of pure strategies and in which a definite set of payments to the n players corresponds to each n-tuple of pure strategies, one strategy being taken by each player.

One such n-tuple counters another if the strategy of each player in the countering n-tuple yields the highest obtainable expectation for its player against the $n-1$ strategies of the other players in the countered n-tuple. A self-countering n-tuple is called **AN EQUILIBRIUM POINT**.

Non-Zero-Sum Game and Nash Equilibrium

Aside:

In a non-zero-sum game, each player chooses a strategy as his best response to other players' strategies.
In a Nash equilibrium, no player will profit from unilaterally changing his strategy.

Non-Zero-Sum Game and Nash Equilibrium

Generalization of zero-sum games:

	Player I	Player II	optimal (s_1^*, s_2^*)
0-sum	$\max_{s_1} R(s_1, s_2)$	$\min_{s_2} R(s_1, s_2)$	"saddle" $R(s_1, s_2^*) \leq R(s_1^*, s_2^*)$, $R(s_1^*, s_2^*) \leq R(s_1^*, s_2)$
0-sum	$\max_{s_1} R(s_1, s_2)$	$\max_{s_2} -R(s_1, s_2)$	$R(s_1^*, s_2^*) \geq R(s_1, s_2^*)$, $-R(s_1^*, s_2^*) \geq -R(s_1^*, s_2)$
non-0-sum	$\max_{s_1} R^1(s_1, s_2)$	$\max_{s_2} R^2(s_1, s_2)$	"equilibrium" $R^1(s_1^*, s_2^*) \geq R^1(s_1, s_2^*)$, $R^2(s_1^*, s_2^*) \geq R^2(s_1^*, s_2)$

Non-Zero-Sum Game and Nash Equilibrium

Simple and understandable example, if there has to be:

go watching *A Beautiful Mind*, Universal Pictures, 2001

(11th Mar. 2009, Columbia University)

Kuhn : Don't learn game theory from the movie. The blonde thing is not a Nash equilibrium!
Odifreddi : How you invented the theory, I mean, the story about the blonde, was it real?
Nash : No!!!
Odifreddi : Did you apply game theory to win Alicia?
Nash : ...Yes...
(followed by 10 min's discussion on personal life and game theory)

Stochastic Differential Games

Martingale Method:
Rewards can be functionals of state process.

- Beneš, 1970, 1971
- M H A Davis, 1979
- Karatzas and Zamfirescu, 2006, 2008

Stochastic Differential Games

BSDE Method:
Identify value of a game to solution to a BSDE, then seek uniqueness and especially existence of solution.

- Bismut, 1970's
- Pardoux and Peng, 1990
- El Karoui, Kapoudjian, Pardoux, Peng, and Quenez, 1997
- Cvitanić and Karatzas, 1996
- Hamadène, Lepeltier, and Peng, 1997

Stochastic Differential Games

PDE Method:
Rewards are functions of state process. Regularity theory by Bensoussan, Frehse, and Friedman. Facilitates numerical computation.

- Bensoussan and Friedman, 1977
- Bensoussan and Frehse, 2000
- H.J. Kushner and P. Dupuis

Our Results

Main results:

- (non-) existence of equilibrium stopping rules
- necessity and sufficiency of Isaacs' condition

Martingale part:

- equilibrium stopping rules, $L \leq U$, $L > U$
- equivalent martingale characterization of Nash equilibrium

BSDE part:

- multi-dim reflective BSDE
- equilibrium stopping rules, $L \leq U$

Mathematical Formulation

Mathematical Formulation

- B is a d-dimensional Brownian motion w.r.t. its generated filtration $\{\mathscr{F}_t\}_{t\geq 0}$ on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
- Change of measure

$$\frac{d\mathbb{P}^{u,v}}{d\mathbb{P}}|\mathscr{F}_t = \exp\{ \int_0^t \sigma^{-1}(s,X)f(s,X,u_s,v_s)dB_s - \frac{1}{2}\int_0^t |\sigma^{-1}(s,X)f(s,X,u_s,v_s)|^2 ds\}, \tag{1}$$

standard $\mathbb{P}^{u,v}$-Brownian motion

$$B_t^{u,v} := B_t - \int_0^t \sigma^{-1}(s,X)f(s,X,u_s,v_s)ds,\ 0 \leq t \leq T. \tag{2}$$

Mathematical Formulation

- State process

$$\begin{aligned} X_t =& X_0 + \int_0^t \sigma(s, X) dB_s, \\ =& X_0 + \int_0^t f(s, X, u_s, v_s) ds + \int_0^t \sigma(s, X) dB_s^{u,v}, \ 0 \le t \le T. \end{aligned} \quad (3)$$

- Hamiltonian

$$\begin{aligned} H_1(t, x, z_1, u, v) &:= z_1 \sigma^{-1}(t, x) f(t, x, u, v) + h_1(t, x, u, v); \\ H_2(t, x, z_2, u, v) &:= z_2 \sigma^{-1}(t, x) f(t, x, u, v) + h_2(t, x, u, v). \end{aligned} \quad (4)$$

Mathematical Formulation

- Admissible controls $u \in \mathscr{U}$ and $v \in \mathscr{V}$.
  $u, v : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ random fields.
- $\tau, \rho \in \mathscr{S}_t$ = set of stopping rules defined on the paths ω, which generate $\{\mathscr{F}_t\}_{t \geq 0}$-stopping times on Ω.
- Strategy: Player I - $(u, \tau(u, v))$; Player II - $(v, \rho(u, v))$.
- Reward processes $R^1(\tau, \rho, u, v)$ and $R^2(\tau, \rho, u, v)$.
- Players' expected reward processes

$$J_t^i(\tau, \rho, u, v) = \mathbb{E}^{u,v}[R_t^i(\tau, \rho, u, v)|\mathscr{F}_t], \; i = 1, 2. \tag{5}$$

Mathematical Formulation

- Nash equilibrium strategies $(u^*, v^*, \tau^*, \rho^*)$

  Find admissible control strategies $u^* \in \mathscr{U}$ and $v^* \in \mathscr{V}$, and stopping rules τ^* and ρ^* in $\mathscr{S}_{t,T}$, that maximize expected rewards.

$$V_1(t) := J_t^1(\tau^*, \rho^*, u^*, v^*) \geq J_t^1(\tau, \rho^*, u, v^*),\ \tau \in \mathscr{S}_{t,T},\ \forall u \in \mathscr{U};$$
$$V_2(t) := J_t^2(\tau^*, \rho^*, u^*, v^*) \geq J_t^2(\tau^*, \rho, u^*, v),\ \forall \rho \in \mathscr{S}_{t,T},\ v \in \mathscr{V}. \tag{6}$$

"no profit from unilaterally changing strategy"

Mathematical Formulation

Analysis, in the spirit of Nash (1949)

For $u^0 \in \mathscr{U}$, $v^0 \in \mathscr{V}$, and $\tau^0, \rho^0 \in \mathscr{S}_{t,T}$, find $(u^1, v^1, \tau^1, \rho^1)$ that counters $(u^0, v^0, \tau^0, \rho^0)$, i.e.

$$
\begin{aligned}
(u^1, \tau^1) &= arg \max_{\tau \in \mathscr{S}_{t,T}} \max_{u \in \mathscr{U}} J_t^1(\tau, \rho^0, u, v^0); \\
(v^1, \rho^1) &= arg \max_{\rho \in \mathscr{S}_{t,T}} \max_{v \in \mathscr{V}} J_t^2(\tau^0, \rho, u^0, v).
\end{aligned}
\tag{7}
$$

The equilibrium $(\tau^*, \rho^*, u^*, v^*)$ is fixed point of the mapping

$$
\Gamma : (\tau^0, \rho^0, u^0, v^0) \mapsto (\tau^1, \rho^1, u^1, v^1) \tag{8}
$$

More about Control Sets

$Z_i^{u,v}(t)$ = instantaneous volatility process of player i's reward process $J_t^i(\tau,\rho,u,v)$, i.e.

$$dJ_t^i(\tau,\rho,u,v) = d \text{ finite variation part} + Z_i^{u,v}(t)dB^{u,v}(t). \tag{9}$$

- Partial observation $u_t = u(t)$, and $v_t = v(t)$.
- Full observation $u_t = u(t,x)$, and $v_t = v(t,x)$.
- Observing volatility

$$u_t = u(t,x,Z_1^{u,v}(t),Z_2^{u,v}(t)), \text{ and } v_t = v(t,x,Z_1^{u,v}(t),Z_2^{u,v}(t)). \tag{10}$$

More about Control Sets

- Why caring about $Z^{u,v}$?
- Risk sensitive control. Sensitive to not only expectation but also variance of the reward.

- Bensoussan, Frehse, and Nagai, 1998
- El Karoui and Hamadène, 2003

Martingale Interpretation

Rewards and Assumptions

$$
\begin{aligned}
R_t^1(\tau,\rho,u,v) &:= \int_t^{\tau\wedge\rho} h_1(s,X,u_s,v_s)ds + L_1(\tau)\mathbb{1}_{\{\tau<\rho\}} + U_1(\rho)\mathbb{1}_{\{\rho\leq\tau<T\}} \\
&\quad + \xi_1\mathbb{1}_{\{\tau\wedge\rho=T\}}; \\
R_t^2(\tau,\rho,u,v) &:= \int_t^{\tau\wedge\rho} h_2(s,X,u_s,v_s)ds + L_2(\rho)\mathbb{1}_{\{\rho<\tau\}} + U_2(\tau)\mathbb{1}_{\{\tau\leq\rho<T\}} \\
&\quad + \xi_2\mathbb{1}_{\{\tau\wedge\rho=T\}}.
\end{aligned} \tag{11}
$$

- boundedness: h, L, U, ξ
- measurabilities: h, L, U, ξ
- continuity: L, U

Equivalent Martingale Characterization

Notations.

$$
\begin{aligned}
Y_1(t;\rho,v) &:= \sup_{\tau\in\mathscr{S}_{t,\rho}} \sup_{u\in\mathscr{U}} J_t^1(\tau,\rho,u,v);\\
Y_2(t;\tau,u) &:= \sup_{\rho\in\mathscr{S}_{t,\tau}} \sup_{v\in\mathscr{V}} J_t^2(\tau,\rho,u,v).
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
V_1(t;\rho,u,v) &:= Y_1(t;\rho,v) + \int_0^t h_1(s,X,u_s,v_s)ds;\\
V_2(t;\tau,u,v) &:= Y_2(t;\tau,u) + \int_0^t h_2(s,X,u_s,v_s)ds.
\end{aligned}
\tag{13}
$$

Equivalent Martingale Characterization

Thm. $(\tau^*, \rho^*, u^*, v^*)$ is an equilibrium point, if and only if the following three conditions hold.

(1) $Y_1(\tau^*; \rho^*, v^*) = L_1(\tau^*)\mathbb{1}_{\{\tau^* < \rho^*\}} + U_1(\rho^*)\mathbb{1}_{\{\rho^* \le \tau^* < T\}} + \xi_1 \mathbb{1}_{\{\tau^* \wedge \rho^* = T\}}$,
and $Y_2(\rho^*; \tau^*, u^*) = L_2(\rho^*)\mathbb{1}_{\{\rho^* < \tau^*\}} + U_2(\tau^*)\mathbb{1}_{\{\tau^* \le \rho^* < T\}} + \xi_2 \mathbb{1}_{\{\tau^* \wedge \rho^* = T\}}$;
(2) $V_1(\cdot \wedge \tau^*; \rho^*, u^*, v^*)$ and $V_2(\cdot \wedge \rho^*; \tau^*, u^*, v^*)$ are $\mathbb{P}^{u^*, v^*}$-martingales;
(3) For every $u \in \mathscr{U}$, $V_1(\cdot \wedge \tau^*; \rho^*, u, v^*)$ is a $\mathbb{P}^{u, v^*}$-supermartingale.
For every $v \in \mathscr{V}$, $V_2(\cdot \wedge \rho^*; \tau^*, u^*, v)$ is a $\mathbb{P}^{u^*, v}$-supermartingale.

Equilibrium Stopping Rules

Def. Generic controls $(u, v) \in \mathscr{U} \times \mathscr{V}$. The equilibrium stopping rules are a pair $(\tau^*, \rho^*) \in \mathscr{S}^2_{t,T}$, such that

$$\begin{aligned} J^1_t(\tau^*, \rho^*, u, v) &\geq J^1_t(\tau, \rho^*, u, v), \ \forall \tau \in \mathscr{S}_{t,T}; \\ J^2_t(\tau^*, \rho^*, u, v) &\geq J^2_t(\tau^*, \rho, u, v), \ \forall \rho \in \mathscr{S}_{t,T}, \end{aligned} \tag{14}$$

Equilibrium Stopping Rules

Notations.

$$\begin{aligned} Y_1(t,u;\rho,v) &:= \sup_{\tau\in\mathscr{S}_{t,T}} J_t^1(\tau,\rho,u,v); \\ Y_2(t,v;\tau,u) &:= \sup_{\rho\in\mathscr{S}_{t,T}} J_t^2(\tau,\rho,u,v). \end{aligned} \tag{15}$$

$$\begin{aligned} Q_1(t,u;\rho,v) &:= Y_1(t,u;\rho,v) + \int_0^t h_1(s,X,u_s,v_s)ds \\ Q_2(t,v;\tau,u) &:= Y_2(t,v;\tau,u) + \int_0^t h_2(s,X,u_s,v_s)ds \end{aligned} \tag{16}$$

Equilibrium Stopping Rules

Lem. (τ^*, ρ^*) is a a pair of equilibrium stopping rules, iff
(1)

$$\begin{aligned} Y_1(\tau^*, u; \rho^*, v) &= L_1(\tau^*)\mathbb{1}_{\{\tau^* < \rho^*\}} + U_1(\rho^*)\mathbb{1}_{\{\rho^* \le \tau^* < T\}} + \xi_1 \mathbb{1}_{\{\tau^* \wedge \rho^* = T\}}, \\ Y_2(\rho^*, v; \tau^*, u) &= L_2(\rho^*)\mathbb{1}_{\{\rho^* < \tau^*\}} + U_2(\tau^*)\mathbb{1}_{\{\tau^* \le \rho^* < T\}} + \xi_2 \mathbb{1}_{\{\tau^* \wedge \rho^* = T\}}; \end{aligned} \tag{17}$$

(2) The stopped supermartingales $Q_1(\cdot \wedge \tau^*, u; \rho^*, v)$ and $Q_2(\cdot \wedge \rho^*, v; \tau^*, u)$ are $\mathbb{P}^{u,v}$-martingales.

Equilibrium Stopping Rules

$L(t) \leq U(t)\mathbb{1}_{\{t<T\}} + \xi\mathbb{1}_{\{t=T\}}$, for all $0 \leq t \leq T$.

If (τ^*, ρ^*) solve the equations

$$\begin{aligned} \tau^* &= \inf\{t \leq s < \rho | Y_1(s, u; \rho^*, v) = L_1(s)\} \wedge \rho^*; \\ \rho^* &= \inf\{t \leq s < \rho | Y_2(s, v; \tau^*, u) = L_2(s)\} \wedge \tau^*, \end{aligned} \tag{18}$$

on first hitting times, then (τ^*, ρ^*) are equilibrium.

Equilibrium Stopping Rules

$L(t) \geq U(t)\mathbb{1}_{\{t<T\}} + \xi\mathbb{1}_{\{t=T\}} + \epsilon$, for all $0 \leq t \leq T$, some $\epsilon > 0$.

If L is uniformly continuous in $\omega \in \Omega$, then equilibrium stopping rules do not exist.

Martingale Structures

Suppose $(\tau^*, \rho^*, u^*, v^*)$ is an equilibrium point.

- Doob-Meyer

$$\begin{aligned} V_1(t;\rho,u,v) =& Y_1(0;\rho,v) - A_1(t;\rho,u,v) + M_1(t;\rho,u,v),\ 0 \le t \le \tau^*; \\ V_2(t;\tau,u,v) =& Y_2(0;\tau,v) - A_2(t;\tau,u,v) + M_2(t;\tau,u,v),\ 0 \le t \le \rho^*. \end{aligned} \tag{19}$$

- Martingale representation

$$\begin{aligned} M_1(t;\rho,u,v) &= \int_0^t Z_1^v(s)dB_s^{u,v}; \\ M_2(t;\tau,u,v) &= \int_0^t Z_2^u(s)dB_s^{u,v}, \end{aligned} \tag{20}$$

Martingale Structures

- Finite variation part

$$
\begin{aligned}
& A_1(t;\tau,u^1,v) - A_1(t;\tau,u^2,v) \\
= & -\int_0^t (H_1(s,X,Z_1(s),u_s^1,v_s) - H_1(s,X,Z_1(s),u_s^2,v_s))ds, \\
& 0 \le t \le \tau^*; \\
& A_2(t;\rho,u,v^1) - A_2(t;\rho,u,v^2) \\
= & -\int_0^t (H_2(s,X,Z_2(s),u_s,v_s^1) - H_2(s,X,Z_2(s),u_s,v_s^2))ds, \\
& 0 \le t \le \rho^*.
\end{aligned}
\tag{21}
$$

Isaacs' Condition

Necessity, stochastic maximum principle

Prop. If $(\tau^*, \rho^*, u^*, v^*)$ is an equilibrium point, then

$$H_1(t, X, Z_1(t), u_t^*, v_t^*) \geq H_1(t, X, Z_1(t), u_t, v_t^*), \text{ for all } 0 \leq t \leq \tau^*, u \in \mathscr{U};$$
$$H_2(t, X, Z_2(t), u_t^*, v_t^*) \geq H_2(t, X, Z_2(t), u_t^*, v_t), \text{ for all } 0 \leq t \leq \rho^*, v \in \mathscr{V}. \tag{22}$$

Isaacs' Condition

Sufficiency

Thm. Let $\tau^*, \rho^* \in \mathscr{S}_{t,T}$ be equilibrium stopping rules. If a pair of controls $(u^*, v^*) \in \mathscr{U} \times \mathscr{V}$ satisfies Isaacs' condition

$$\begin{aligned} H_1(t, x, z_1, u^*_t, v^*_t) &\geq H_1(t, x, z_1, u_t, v^*_t); \\ H_2(t, x, z_2, u^*_t, v^*_t) &\geq H_2(t, x, z_2, u^*_t, v_t), \end{aligned} \tag{23}$$

for all $0 \leq t \leq T$, $u \in \mathscr{U}$, and $v \in \mathscr{V}$, then u^*, v^* are equilibrium controls.

BSDE Approach

Each Player's Reward Terminated by Himself

Game 2.1

$$R_t^1(\tau,\rho,u,v) := \int_t^\tau h_1(s,X,u_s,v_s)ds + L_1(\tau)\mathbb{1}_{\{\tau<T\}} + \eta_1\mathbb{1}_{\{\tau=T\}};$$
$$R_t^2(\tau,\rho,u,v) := \int_t^\rho h_2(s,X,u_s,v_s)ds + L_2(\rho)\mathbb{1}_{\{\rho<T\}} + \eta_2\mathbb{1}_{\{\rho=T\}}. \tag{24}$$

$L_1 \le \eta_1$, $L_2 \le \eta_2$, a.s.

Assumption A 2.1 (Isaac's condition) There exist admissible controls $(u^*, v^*) \in \mathscr{U} \times \mathscr{V}$, such that $\forall t \in [0,T]$, $\forall u \in \mathscr{U}$, $\forall v \in \mathscr{V}$,

$$H_1(t,x,z_1,(u^*,v^*)(t,x,z_1,z_2)) \ge H_1(t,x,z_1,u(t,x,\cdot,\cdot),v^*(t,x,z_1,z_2));$$
$$H_2(t,x,z_2,(u^*,v^*)(t,x,z_1,z_2)) \ge H_2(t,x,z_2,u^*(t,x,z_1,z_2),v(t,x,\cdot,\cdot)). \tag{25}$$

Each Player's Reward Terminated by Himself

Thm 2.1 Let (Y, Z, K) be solution to reflective BSDE

$$\begin{cases} Y(t) = \eta + \int_t^T H(s, X, Z(s), u^*, v^*)ds - \int_t^T Z(s)dB_s + K(T) - K(t); \\ Y(t) \geq L(t), 0 \leq t \leq T; \int_0^T (Y(t) - L(t))dK_i(t) = 0. \end{cases} \tag{26}$$

Optimal stopping rules

$$\begin{aligned} \tau^* &:= \inf\{s \in [t, T] : Y_1(s) \leq L_1(s)\} \wedge T; \\ \rho^* &:= \inf\{s \in [t, T] : Y_2(s) \leq L_2(s)\} \wedge T. \end{aligned} \tag{27}$$

$(\tau^*, \rho^*, u^*, v^*)$ is optimal for Game 2.1. Further more, $V_i(t) = Y_i(t)$, $i = 1, 2$.

Game with Interactive Stoppings

Game 2.2

$$R_t^1(\tau,\rho,u,v) := \int_t^{\tau\wedge\rho} h_1(s,X,u_s,v_s)ds + L_1(\tau)\mathbb{1}_{\{\tau<\rho\}} + U_1(\rho)\mathbb{1}_{\{\rho\le\tau<T\}} + \xi_1\mathbb{1}_{\{\tau\wedge\rho=T\}};$$

$$R_t^2(\tau,\rho,u,v) := \int_t^{\tau\wedge\rho} h_2(s,X,u_s,v_s)ds + L_2(\rho)\mathbb{1}_{\{\rho<\tau\}} + U_2(\tau)\mathbb{1}_{\{\tau\le\rho<T\}} + \xi_2\mathbb{1}_{\{\tau\wedge\rho=T\}}. \quad (28)$$

Assumption A 2.2 (Isaac's condition) There exist admissible controls $(u^*,v^*)\in\mathscr{U}\times\mathscr{V}$, such that

$$H_1(t,x,z_1,(u^*,v^*)(t,x)) \ge H_1(t,x,z_1,u(t,x),v^*(t,x)),\ \forall t\in[0,T],\ \forall u\in\mathscr{U}$$
$$H_2(t,x,z_2,(u^*,v^*)(t,x)) \ge H_2(t,x,z_2,u^*(t,x),v(t,x)),\ \forall t\in[0,T],\ \forall v\in\mathscr{V}$$

Game with Interactive Stoppings

Assumption A 2.3

(1) For $i = 1, 2$, the reward processes $U_i(\cdot)$ redefined as

$$U_i(t) = \begin{cases} U_i(t), 0 \leq t < T; \\ \xi_i, t = T, \end{cases} \tag{30}$$

are increasing processes. $L(t)_i \leq U_i(t) \leq \xi$, a.s. ("patience pays")
(2) Both reward processes $\{U(t)\}_{0 \leq t \leq T}$ and $\{L(t)\}_{0 \leq t \leq T}$ are right continuous in time t.
$[0, T] \times \Omega$.
(3) $h_i \geq -c$, $i = 1, 2$.

Game with Interactive Stoppings

Thm 2.2 Under assumptions A 2.2 and A 2.3, then there exists an equilibrium point $(\tau^*, \rho^*, u^*, v^*)$ of Game 2.2.

Game with Interactive Stoppings

Thm 2.3 (Associated BSDE)

$$\begin{cases} Y_i(t) = \xi_i + \int_t^T H_i(s, X, Z_i(s), (u,v)(t, X, Z_1(s), Z_2(s))ds \\ \qquad - \int_t^T Z_i(s)dB_s + K_i(T) - K_i(t) + N_i(t,T), \\ Y_i(t) \geq L_i(t), 0 \leq t \leq T; \ \int_0^T (Y_i(t) - L_i(t))dK_i(t) = 0; \ i = 1, 2, \end{cases} \tag{31}$$

where

$$N_i(t,T) := \sum_{t \leq s \leq T} (U_i(s) - Y_i(s)) \mathbb{1}_{\{Y_j(s) = L_j(s)\}}, \ i, j = 1, 2. \tag{32}$$

(being kicked up to U, when the other player drops down to L)

Game with Interactive Stoppings

Optimal stopping times

$$\begin{aligned} \tau^* &:= \tau_t^*(u,v) := \inf\{s \in [t,T] : Y_1^{u,v}(s) \leq L_1(s)\} \wedge T; \\ \rho^* &:= \rho_t^*(u,v) := \inf\{s \in [t,T] : Y_2^{u,v}(s) \leq L_2(s)\} \wedge T. \end{aligned} \tag{33}$$

Multi-Dim Reflective BSDE

Lipschitz Growth

m-dim reflective BSDE

$$\begin{cases} Y_1(t) = \xi_1 + \int_t^T g_1(s, Y(s), Z(s))ds - \int_t^T Z_1(s)' dB_s + K_1(T) - K_1(t) \\ Y_1(t) \geq L_1(t), 0 \leq t \leq T, \int_0^T (Y_1(t) - L_1(t))' dK_1(t) = 0, \\ \ldots \\ Y_m(t) = \xi_m + \int_t^T g_m(s, Y(s), Z(s))ds - \int_t^T Z_m(s)' dB_s + K_m(T) - K_m \\ Y_m(t) \geq L_m(t), 0 \leq t \leq T, \int_0^T (Y_m(t) - L_m(t))' dK_m(t) = 0. \end{cases} \tag{34}$$

Lipschitz Growth

Seek solution (Y, Z, K) in the spaces

$$
\begin{aligned}
& Y = (Y_1, \cdots, Y_m)' \in \mathbb{M}^2(m; 0, T) \\
:= & \{m\text{-dimensional predictable process } \phi \text{ s.t. } \mathbb{E}[\sup_{[0,T]} \phi_t^2] \leq \infty\}; \\
& Z = (Z_1, \cdots, Z_m)' \in \mathbb{L}^2(m \times d; 0, T) \\
:= & \{m \times d\text{-dimensional predictable process } \phi \text{ s.t. } \mathbb{E}[\int_0^T \phi_t^2 dt] \leq \infty\}; \\
& K = (K_1, \cdots, K_m)' = \text{ continuous, increasing process in } \mathbb{M}^2(m; 0, T).
\end{aligned}
\tag{35}
$$

Lipschitz Growth

Assumption A 3.1

(1) The random field

$$g = (g_1, \cdots, g_m)' : [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}^m \qquad (36)$$

is uniformly Lipschitz in y and z, i.e. there exists a constant $b > 0$, such that

$$|g(t, y, z) - g(t, \bar{y}, \bar{z})| \le b(\|y - \bar{y}\| + \|z - \bar{z}\|), \forall t \in [0, T]. \qquad (37)$$

Further more,

$$\mathbb{E}[\int_0^T g(t, 0, 0)^2 dt] < \infty. \qquad (38)$$

(2) The random variable ξ is $\mathscr{F}_T$-measurable and square-integrable. The lower reflective boundary L is progressively measurable, and satisfy $\mathbb{E}[\sup_{[0,T]} L^+(t)^2] < \infty$. $L \le \xi$, $\mathbb{P}$-a.s.

Lipschitz Growth

Results:

- existence and uniqueness of solution, via contraction method
- 1-dim Comparison Theorem (EKPPQ, 1997)
- continuous dependency property

Linear Growth, Markovian System

l(elle)-dim forward equation

$$\begin{cases} X^{t,x}(s) = x, \, 0 \leq s \leq t; \\ dX^{t,x}(s) = \sigma(s, X^{t,x}(s))' dB_s, \, t < s \leq T. \end{cases} \quad (39)$$

m-dim backward equation

$$\begin{cases} Y^{t,x}(s) = \xi(X^{t,x}(T)) + \int_s^T g_i(r, X^{t,x}(r), Y^{t,x}(r), Z^{t,x}(r)) dr \\ \qquad - \int_s^T Z^{t,x}(r)' dB_r + K^{t,x}(T) - K^{t,x}(s); \\ Y^{t,x}(s) \geq L(s, X^{t,x}(s)), \, t \leq s \leq T, \, \int_t^T (Y^{t,x}(s) - L(s, X^{t,x}(s)))' dK^{t,x}(s) \end{cases} \quad (40)$$

Linear Growth, Markovian System

Assumption A 4.1

(1) $g : [0, T] \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^m$ is measurable, and for all $(t, x, y, z) \in [0, T] \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$, $|g(t, x, y, z)| \leq b(1 + |x|^p + |y| + |z|)$, for some positive constant b;

(2) for every fixed $(t, x) \in [0, T] \times \mathbb{R}$, $g(t, x, \cdot, \cdot)$ is continuous.

(3) $\mathbb{E}[\xi(X(T))^2] < \infty$; $\mathbb{E}[\sup_{[0,T]} L^+(t, X(t))^2] < \infty$. $L \leq \xi$, $\mathbb{P}$-a.s.

Linear Growth, Markovian System

Results

- existence of solution, via Lipschitz approximation
- 1-dim Comparison Theorem
- continuous dependency property

THAT'S ALL

THANK YOU